Test 2 **Theory of Probability** **April 4, 2019**

MATH 464 **Dr. Abdul-Rahman**

Name: ______________________ **Signature:** ______________________

1. [15 points] A random variable X has the following moment generating function

$$\mathbb{M}_X(t) = c + \frac{3}{4}e^t + \frac{1}{8}e^{2t}$$

   (a) Write the probability mass function of X, i.e., f_X in a tabular form.

   (b) Find a formula for the n-th moment of X, i.e., $\mathbb{E}(X^n)$ for $n = 1, 2, \ldots$

   (c) Find $\mathrm{Var}(X^2)$.

2. [15 points] Let X be a continuous random variable with density function

$$f_X(x) = \begin{cases} \frac{c}{1+x^2}, & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c.

(b) Find the distribution function of X, i.e., $F_X(x)$.

3. [15 points] Let X and Y be two independent random variables with the following probability mass functions

$$f_X(2) = f_X(4) = f_Y(1) = f_Y(2) = \frac{1}{2},$$

Let $U := XY$ and $W := X/Y$.

(a) Find $\mathbb{E}(U)$ and $\mathbb{E}(W)$.

(b) Find $\mathrm{Var}(UW)$.

(c) Find $\mathbb{P}(X = 2, U = 2)$ and $\mathbb{P}(W = 2, U = 4)$.

4. [15 points] Let $X \sim \text{Exp}(\lambda)$, for $\lambda > 0$. Find the distribution and the density of $Y = \sqrt{X+1} - 1$.

5. [15 points] Suppose $X_1, \ldots, X_n$ are independent random variables with $\mathbb{E}(X_j) = \mu$ and $\mathrm{Var}(X_j) = \sigma^2$ for all $j = 1, \ldots, n$. Let $Y_k = X_1 + \ldots + X_k$ for $1 \leq k \leq n$.

   (a) Find $\mathbb{E}(Y_n^2)$.

   (b) Find $\mathbb{E}\left(\sum_{k=1}^{n} Y_k\right)$.

6. [15 points] If X is a discrete uniform random variable on $\{0, 1, \ldots, a\}$, i.e.,

$$\mathbb{P}(X = k) = \frac{1}{1+a}, \text{ for } k = 0, 1, 2, \ldots, a,$$

show that X has moment generating function

$$\mathbb{M}_X(t) = \frac{1 - e^{t(a+1)}}{(1+a)(1-e^t)}, \ t \in \mathbb{R}$$

7. [15 points] Let X and Y be independent discrete random variables. Given that

$$\mathbb{E}(X^n) = 3^{n-1}, \text{ and } \mathbb{E}(Y^n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases} \quad \text{for } n = 1, 2, \ldots$$

(a) Give an example of such random variables X and Y.

(b) Let $Z = 2X + Y$. Find the mean and variance of Z.

(c) Let $W = Y^2 - 2YX$. Find the mean and variance of W.

1- Discrete Random variables (for $0 \le p \le 1$, we use $q := 1 - p$)

Bernoulli: $X \sim \text{Bernoulli}(p)$

$$\mathbb{P}(X = 1) = p \quad \text{and} \quad P(X = 0) = q\,.$$
$$\mathbb{E}(X) = p, \quad \text{Var}(X) = pq, \;\; M_X(t) = q + pe^t.$$

Binomial: $X \sim \text{Binomial}(n, p)$, $n \in \mathbb{N}$.

$$\mathbb{P}(X = k) = \binom{n}{k} q^{n-k} p^k, \quad k = 0, 1, \ldots, n.$$
$$\mathbb{E}(X) = np, \quad \text{Var}(X) = npq, \;\; M_X(t) = (q + pe^t)^n.$$

Geometric: $X \sim \text{Geometric}(p)$

$$\mathbb{P}(X = k) = q^{k-1} p, \quad k = 1, 2, \ldots$$
$$\mathbb{E}(X) = \frac{1}{p}, \quad \text{Var}(X) = \frac{q}{p^2}, \;\; M_X(t) = \frac{pe^t}{1 - qe^t}.$$

Poisson: $X \sim \text{Poisson}(\lambda)$, $\lambda > 0$.

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, \ldots$$
$$\mathbb{E}(X) = \lambda, \quad \text{Var}(X) = \lambda, \;\; M_X(t) = e^{\lambda(e^t - 1)}.$$

Negative Binomial: $X \sim \text{NB}(n, p)$, $n \in \mathbb{N}$.

$$\mathbb{P}(X = k) = \binom{k-1}{n-1} p^n q^{k-n}, \quad k = n, n+1, \ldots$$
$$\mathbb{E}(X) = \frac{n}{p}, \quad \text{Var}(X) = \frac{nq}{p^2}, \;\; M_X(t) = \left(\frac{pe^t}{1 - qe^t}\right)^n.$$

2- Continuous Random variables

Uniform: $X \sim U(a, b)$, where $a < b$.

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0 & \text{otherwise} \end{cases}$$
$$\mathbb{E}(X) = \frac{a+b}{2}, \quad \text{Var}(X) = \frac{(b-a)^2}{12}, \quad M_X(t) = \frac{e^{bt} - e^{at}}{t(b-a)}, \; t \neq 0.$$

Exponential: $X \sim \text{Exp}(\lambda)$, where $\lambda > 0$.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$\mathbb{E}(X) = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}, \quad M_X(t) = \frac{\lambda}{\lambda - t}, \; t < \lambda.$$

Normal: $X \sim N(\mu, \sigma^2)$, where $\mu, \sigma \in \mathbb{R}$.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty.$$
$$\mathbb{E}(X) = \mu, \quad \text{Var}(X) = \sigma^2, \quad M_X(t) = e^{\mu t + \frac{(\sigma t)^2}{2}}.$$

Cauchy:

$$f_X(x) = \frac{1}{\pi(1 + x^2)}, \quad -\infty < x < \infty.$$
$$\mathbb{E}(X) = \text{undefined}, \quad \text{Var}(X) = \text{undefined}.$$

Gamma: $X \sim \Gamma(\omega, \lambda)$, where $\omega, \lambda > 0$.

$$f_X(x) = \begin{cases} \frac{1}{\Gamma(\omega)} \lambda^\omega x^{\omega-1} e^{-\lambda x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}, \quad \Gamma(\omega) = \int_0^\infty x^{\omega-1} e^{-x} dx.$$
$$\mathbb{E}(X) = \frac{\omega}{\lambda}, \quad \text{Var}(X) = \frac{\omega}{\lambda^2}, \quad M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^\omega, \; t < \lambda.$$